Math 211 - Bonus Exercise 6 (please discuss on Forum)

1) Give examples (with proof) of an abelian group G such that

- G is an infinite p-group for some prime p
- G is an infinite torsion group, and for any natural number n, there exists an element of G of order n
- any finite group is isomorphic to some subgroup of G
- $G \cong G \times G$

(note: give one group per each of the 4 properties above; no single group will satisfy all of them).

2) Construct a one-to-one correspondence between finite abelian groups of the form

$$\mathbb{Z}/p_1^{d_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{d_k}\mathbb{Z}$$

(where the prime powers $p_i^{d_i}$ are the elementary divisors) and finite abelian groups of the form

$$\mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_s\mathbb{Z}$$

(where the natural numbers $n_1, \dots, n_s \geq 2$ satisfy $n_1 | n_2 | \dots | n_s$ and are call invariant factors). What is the meaning of the largest invariant factor n_s in purely group-theoretic terms?

3) Compute the number of abelian groups of order n (up to isomorphism) in terms of the exponents of the prime factors of n.

4) Let G be a finite abelian group, and consider the homomorphism

$$f : G \to G, \qquad g \mapsto pg$$

for a fixed prime number p. Prove that $G/\text{Im } f \cong \text{Ker } f$ (which is different from the usual formula).

5) For any group G, its dual is defined as

$$\widehat{G} = \Big\{ \text{homomorphisms } G \to \mathbb{Q}/\mathbb{Z} \Big\}$$

where the right-hand side is made into a group with respect to pointwise addition: $(f + f')(g) = f(g) + f'(g), \forall g \in G$. Prove that $\widehat{G}$ is an abelian group, which is isomorphic to G itself if G is finite.